Detecting Chaos in Autonomous Hamiltonian Systems and Symplectic Maps

Charalampos (Haris) Skokos

Research Center for Astronomy and Applied Mathematics,

Academy of Athens, Athens, Greece

and

Center for Research and Applications of Nonlinear Systems (CRANS),
University of Patras, Patras, Greece

E-mail: hskokos@cc.uoa.gr

URL: http://www.math.upatras.gr/~skokos/

Outline

- Hamiltonian systems – Poincaré Surface of Section (PSS) – Symplectic maps

- Chaos Detection Tools
  - ✓ Maximal Lyapunov characteristic exponent
  - ✓ Other methods
  - ✓ Smaller Alignment Index (SALI)

Autonomous Hamiltonian systems

Consider an N degree of freedom autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\underbrace{q_1, q_2, \ldots, q_N}_{\text{positions}}, \underbrace{p_1, p_2, \ldots, p_N}_{\text{momenta}})$$

The time evolution of an orbit (trajectory) with initial condition

$$P(0)=(q_1(0), q_2(0), \ldots, q_N(0), p_1(0), p_2(0), \ldots, p_N(0))$$

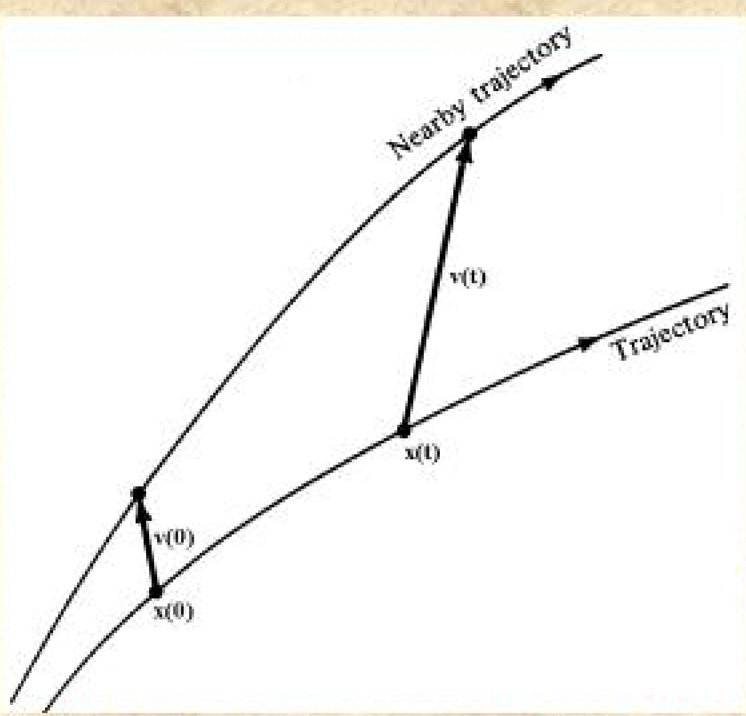
is governed by the Hamilton's equations of motion

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad , \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

Variational Equations

We use the notation $\mathbf{x} = (q_1,q_2,\ldots,q_N,p_1,p_2,\ldots,p_N)^T$. The **deviation vector** from a given orbit is denoted by

$$\mathbf{v} = (dx_1, dx_2,\ldots, dx_n)^T \text{ , with } n=2N$$



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

The time evolution of v is given by the so-called **variational equations:**

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J}\cdot\mathbf{P}\cdot\mathbf{v}$$

where

$$\mathbf{J} = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix},\ \mathbf{P}_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j}\quad i,j=1,2,\ldots,n$$

Example (Hénon-Heils system)

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3$$

Hamilton's equations of motion:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \Rightarrow \begin{cases} \dot{x} = p_x \\ \dot{y} = p_y \\ \boxed{\dot{p}_x = -x - 2xy} \\ \dot{p}_y = -y - x^2 + y^2 \end{cases}$$

In order to get the variational equations we linearize the above equations by substituting x, y, px, py with $x+v_1$, $y+v_2$, p_x+v_3, p_y+v_4 where $v=(v_1,v_2,v_3,v_4)$ is the deviation vector. So we get:

$$\dot{p}_x + \dot{v}_3 = -x - v_1 - 2(x + v_1)(y + v_2) \Rightarrow$$

$$\dot{p}_x + \dot{v}_3 = -x - v_1 - 2xy - 2xv_2 - 2yv_1 - 2v_1 v_2 \Rightarrow$$

$$\dot{v}_3 = -v_1 - 2yv_1 - 2xv_2$$

Example (Hénon-Heils system)

Variational equations: $\frac{dv}{dt} = -J \cdot P \cdot v$

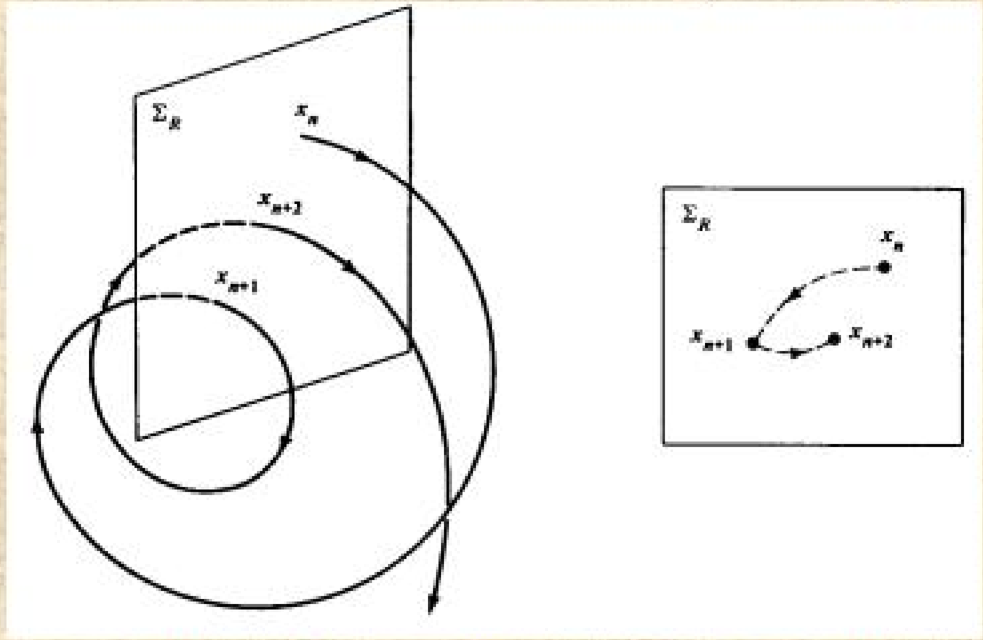
$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1+2y & 2x & 0 & 0 \\ 2x & 1-2y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$\begin{aligned} \dot{v}_1 &= v_3 \\ \dot{v}_2 &= v_4 \\ \dot{v}_3 &= -v_1 - 2xv_2 - 2yv_1 \\ \dot{v}_4 &= -v_2 - 2xv_1 + 2yv_2 \end{aligned} \quad + \quad \begin{aligned} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= -y - x^2 + y^2 \end{aligned}$$

Complete set of equations

Poincaré Surface of Section (PSS)

We can constrain the study of an N+1 degree of freedom Hamiltonian system to a 2N-dimensional subspace of the general phase space.



Lieberman & Lichtenberg, 1992, *Regular and Chaotic Dynamics*, Springer.

In general we can assume a PSS of the form q_{N+1}=constant. Then only variables $q_1,q_2,\ldots,q_N,p_1,p_2,\ldots,p_N$ are needed to describe the evolution of an orbit on the PSS, since p_{N+1} can be found from the Hamiltonian.

In this sense an N+1 degree of freedom Hamiltonian system corresponds to a 2N-dimensional symplectic map.

Symplectic Maps

Consider an n-dimensional symplectic map T. In this case we have discrete time.

The evolution of an orbit with initial condition

$$P(0)=(x_1(0), x_2(0),\ldots,x_n(0))$$

is governed by the equations of map T

$$P(i+1)=T\,P(i)\ ,\ i=0,1,2,\ldots$$

The evolution of an initial deviation vector

$$v(0) = (dx_1(0), dx_2(0),\ldots, dx_n(0))$$

is given by the corresponding tangent map

$$v(i+1) = \left.\frac{\partial T}{\partial P}\right|_i \cdot v(i)\ ,\ i = 0,1,2,\ldots$$

Example – 2D map

Equations of the map:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{array}{lcl} x_1' & = & x_1 + x_2 \\ x_2' & = & x_2 - \nu \sin(x_1 + x_2) \end{array} \quad (\text{mod } 2\pi)$$

Tangent map:

$$v(i+1) = \left.\frac{\partial T}{\partial P}\right|_i \cdot v(i)$$

$$\begin{pmatrix} dx_1' \\ dx_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\nu\cos(x_1 + x_2) & 1 - \nu\cos(x_1 + x_2) \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix}$$

Lyapunov Exponents

Roughly speaking, the Lyapunov exponents of a given orbit characterize the **mean exponential rate of divergence** of trajectories surrounding it.

Consider an orbit in an M-dimensional phase space with **initial condition x(0)** and an **initial deviation vector from it v(0).** Then the mean exponential rate of divergence is:

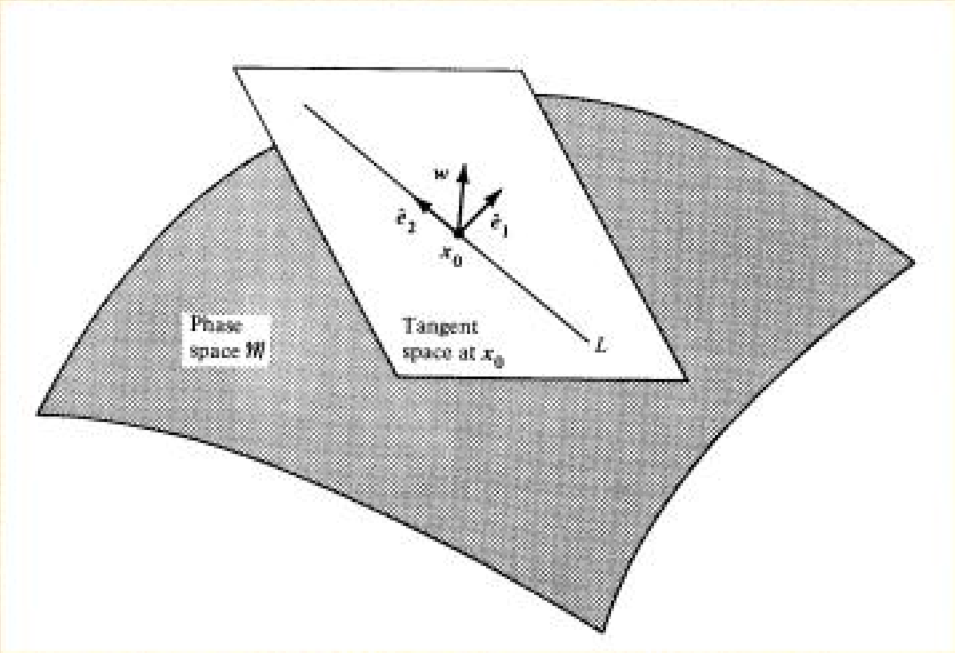
$$\sigma(x(0), v(0)) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

Lyapunov Exponents

There exists an M-dimensional basis $\{\hat{e}_i\}$ of v such that for any v, σ takes on one of the M (possibly nondistinct) values

$$\sigma_i(x(0)) = \sigma(x(0), \hat{e}_i)$$

which are the Lyapunov exponents.



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

In autonomous Hamiltonian systems the M exponents are ordered in pairs of opposite sign numbers and two of them are 0.

Computation of the Maximal Lyapunov Exponent

Due to the exponential growth of v(t) (and of d(t)=||v(t)||) we renormalize v(t) from time to time.

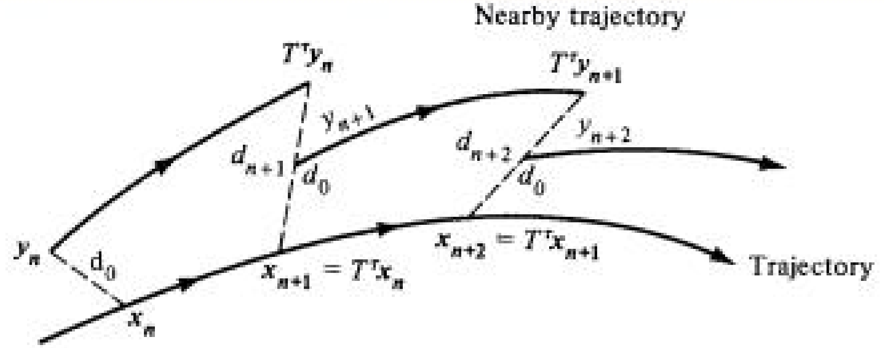


Figure 5.6. Numerical calculation of the maximal Liapunov characteristic exponent. Here $y = x + v$ and τ is a finite interval of time (after Benettin *et al.*, 1976).

Then the Maximal Lyapunov exponent is computed as

$$\sigma_1 = \lim_{n \to \infty} \frac{1}{n\tau} \sum_{i=1}^{n} \ln d_i$$

Maximal Lyapunov Exponent

$\sigma_1 = 0 \rightarrow$ **Ordered motion**
$\sigma_1 \neq 0 \rightarrow$ **Chaotic motion**

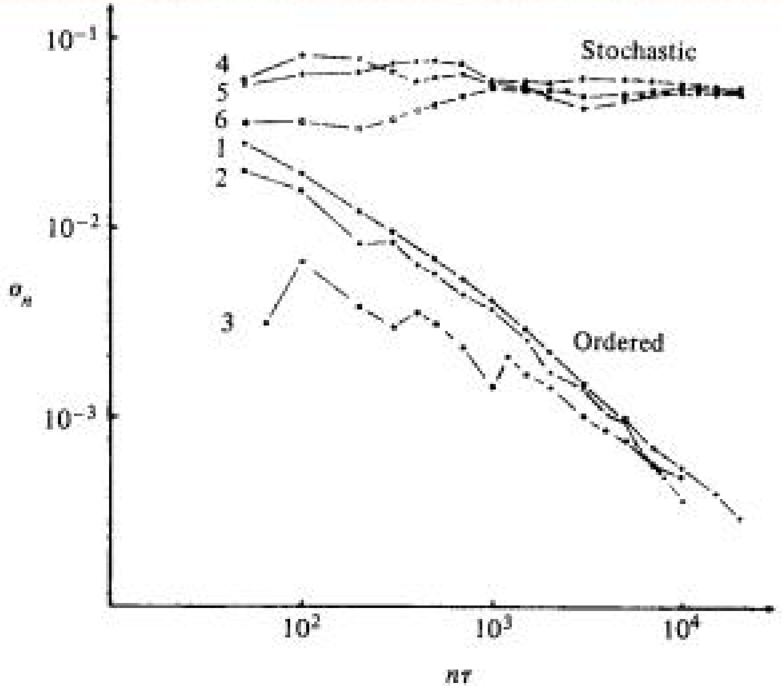


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al.*, 1976).

Benettin et al. (1980) proposed an algorithm for the computation of all Lyapunov exponents

Other methods

- **Frequency Analysis** (Laskar J., 1990, Icarus, 88, 257 – Laskar et al., 1992, Physica D, 56, 253 – Papaphilippou Y. & Laskar J., Astron. Astroph, 1996, 307, 427 & 1998, 329, 451)
- **Dynamical Spectra** (Froeschlé et al., 1993, Cel. Mech., 56, 307 – Voglis N. & Contopoulos G., 1994, J. Phys. A, 27, 4899– Voglis et al., 1999, Cel. Mech., 73, 211 & 1998, Phys. Rev. E, 57, 372)
- **Fast Lyapunov Indicator (FLI)** (Froeschlé et al., 1997, Cel. Mech., 67, 41 – Froeschlé et al., 1997, Planet. Space Sci., 45, 881)
- **0-1 test** (Gottwald G. A. & Melbourne I., 2004, Proc. R. Soc. Lond. A, 460, 603)

The Smaller Alignment Index (SALI) method

Work in collaboration with

- Chris Antonopoulos
- Thanos Manos
- Tassos Bountis
- Michael Vrahatis

Papers

- Skokos Ch. (2001) J. Phys. A, 34, 10029.
- Skokos Ch., Antonopoulos Ch., Bountis T. C. & Vrahatis M. N. (2003) Prog. Theor. Phys. Supp., 150, 439.
- Skokos Ch., Antonopoulos Ch., Bountis T. C. & Vrahatis M. N. (2004) J. Phys. A, 37, 6269.

Definition of the Smaller Alignment Index (SALI)

Consider the n-dimensional phase space of a conservative dynamical system (a symplectic map a Hamiltonian flow).

An orbit in that space with initial condition :

$$P(0)=(x_1(0), x_2(0),\ldots,x_n(0))$$

and a deviation vector

$$v(0)=(dx_1(0), dx_2(0),\ldots, dx_n(0))$$

The evolution in time (in maps the time is discrete and is equal to the number N of the iterations) of a deviation vector is defined by:

- **the variational equations (for Hamiltonian flows) and**
- **the equations of the tangent map (for mappings)**

Definition of the SALI

We follow the evolution in time of two different initial deviation vectors (e.g. $v_1(0)$, $v_2(0)$), and define SALI (Skokos Ch., 2001, J. Phys. A, 34, 10029) as:

$$\mathrm{SALI}(t) = \min\left\{ \left\| \frac{v_1(t)}{\|v_1(t)\|} + \frac{v_2(t)}{\|v_2(t)\|} \right\|, \left\| \frac{v_1(t)}{\|v_1(t)\|} - \frac{v_2(t)}{\|v_2(t)\|} \right\| \right\}$$

When the two vectors tend to coincide or become opposite

$$\mathrm{SALI}(t) \rightarrow 0$$

Behavior of the SALI

2D maps

SALI→0 both for ordered and chaotic orbits

following, however, completely different time rates which allows us to distinguish between the two cases.

Hamiltonian flows and multidimensional maps

SALI→0 for chaotic orbits

SALI→constant ≠ 0 for ordered orbits

Behavior of the SALI

Hamiltonian flows and multidimensional maps

The ordered motion occurs on a torus and two different initial deviation vectors become tangent to different directions on the torus.

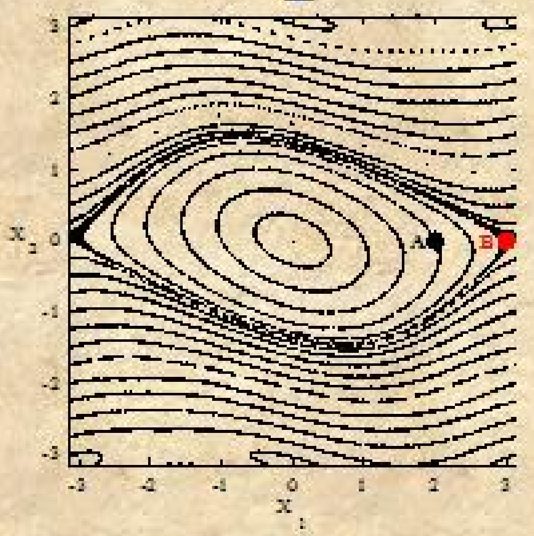
In chaotic cases two initially different deviation vectors tend to coincide to the direction defined by the most unstable nearby manifold.

2D maps

Any two deviation vectors tend to coincide or become opposite for ordered and chaotic orbits.

Applications – 2D map

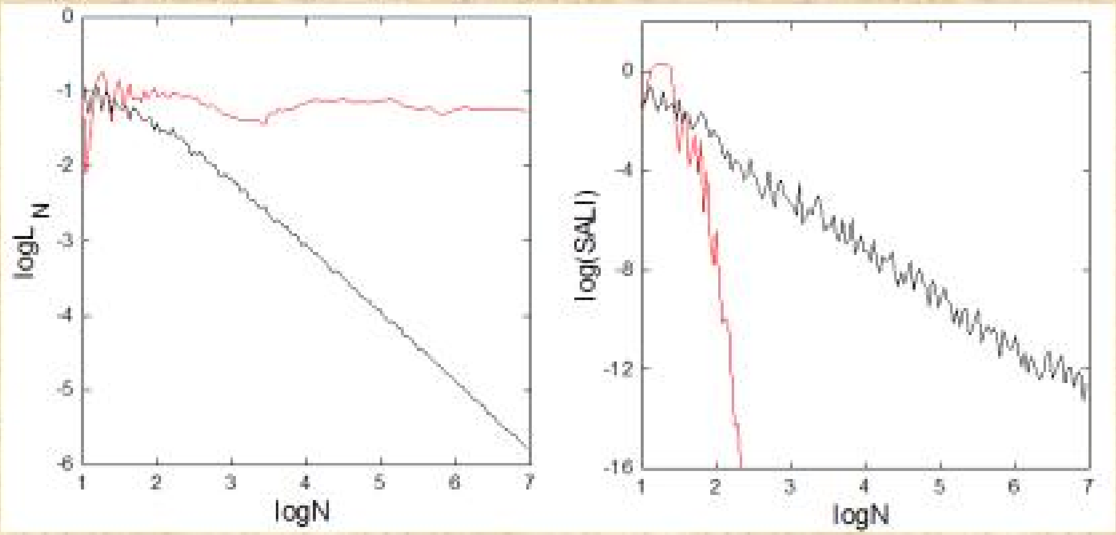
$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \sin(x_1 + x_2) \end{aligned} \quad (\text{mod } 2\pi)$$



For ν=0.5 we consider the orbits:

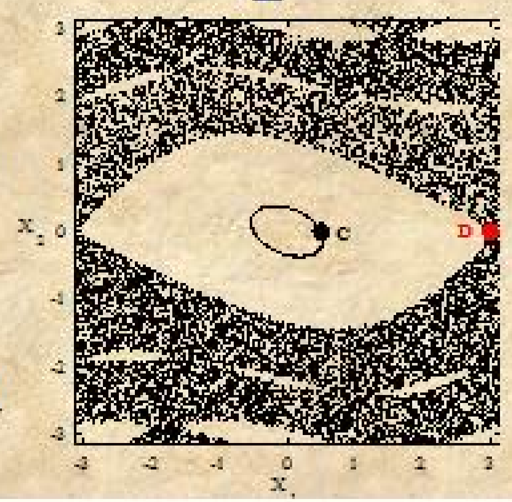
ordered orbit A *with initial conditions $x_1=2$, $x_2=0$.*

chaotic orbit B *with initial conditions $x_1=3$, $x_2=0$.*



Applications – 4D map

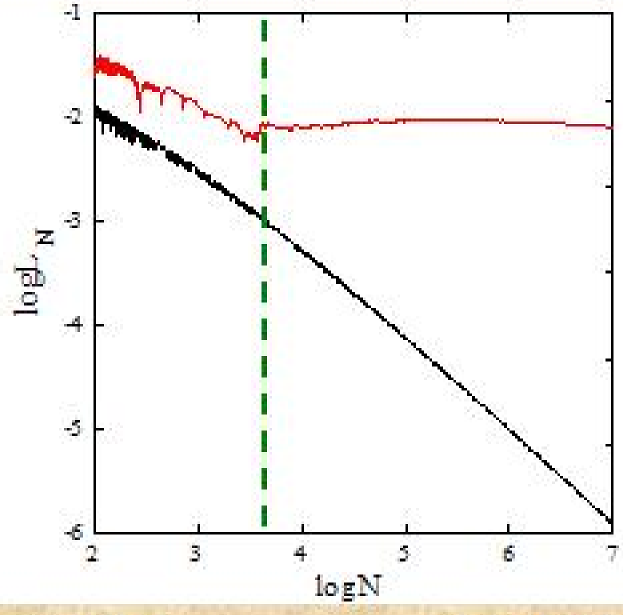
$$
\begin{aligned}
x_1' &= x_1 + x_2 \\
x_2' &= x_2 - \nu \sin(x_1 + x_2) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)] \\
x_3' &= x_3 + x_4 \\
x_4' &= x_4 - \kappa \sin(x_3 + x_4) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)]
\end{aligned}
\qquad (\text{mod } 2\pi)
$$

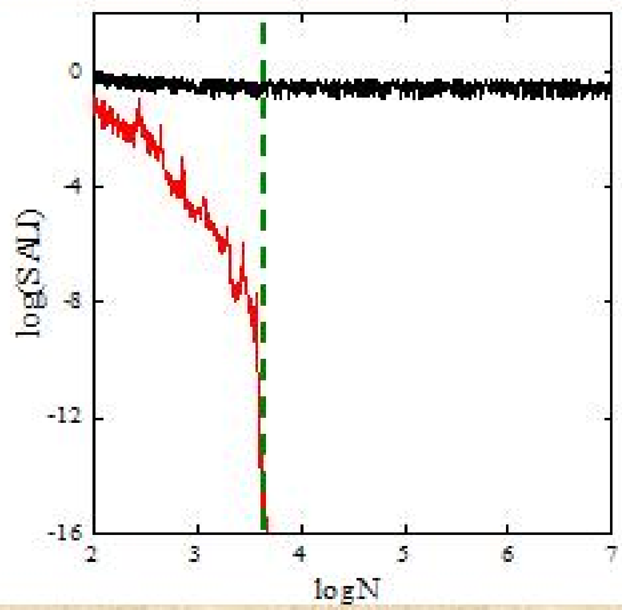


For $\nu=0.5$, $\kappa=0.1$, $\mu=0.1$ we consider the orbits:

ordered orbit C *with initial conditions $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.*

chaotic orbit D *with initial conditions $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.*





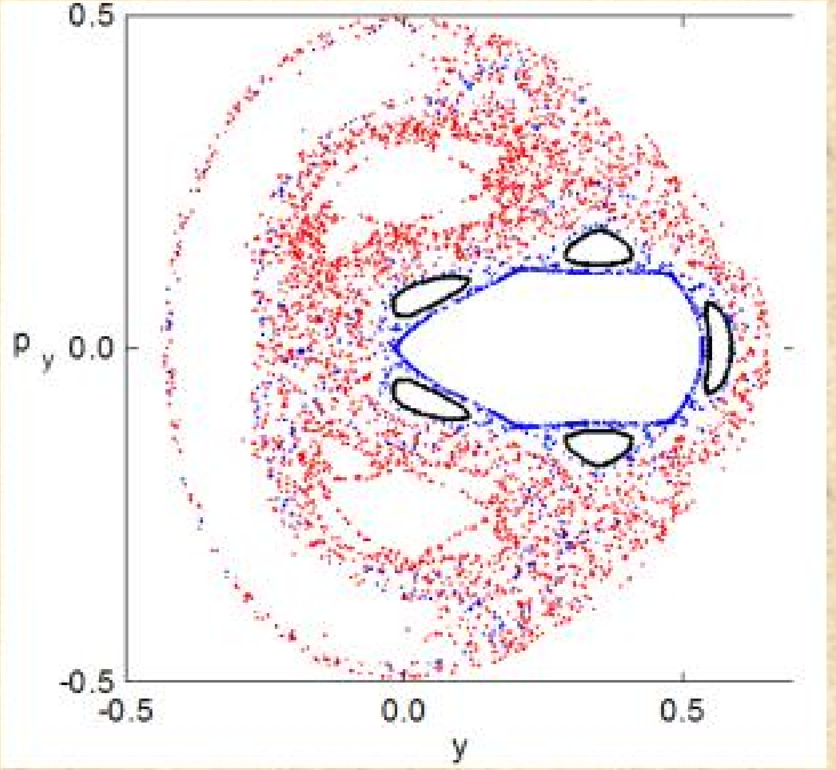
Applications – Hénon-Heils system

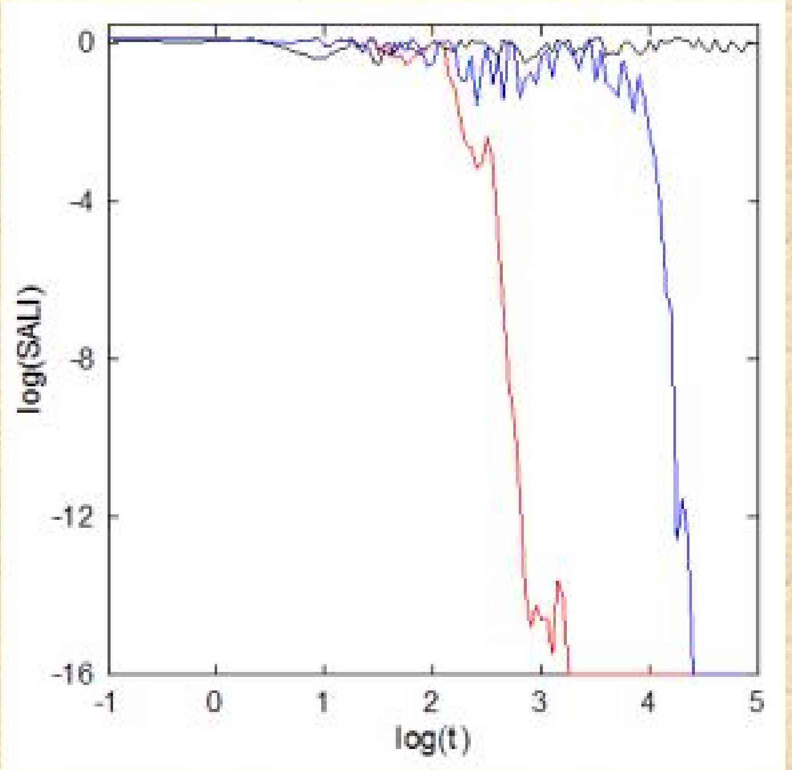
For E=1/8 we consider the orbits with initial conditions:

Ordered orbit, x=0, y=0.55, p_x=0.2417, p_y=0

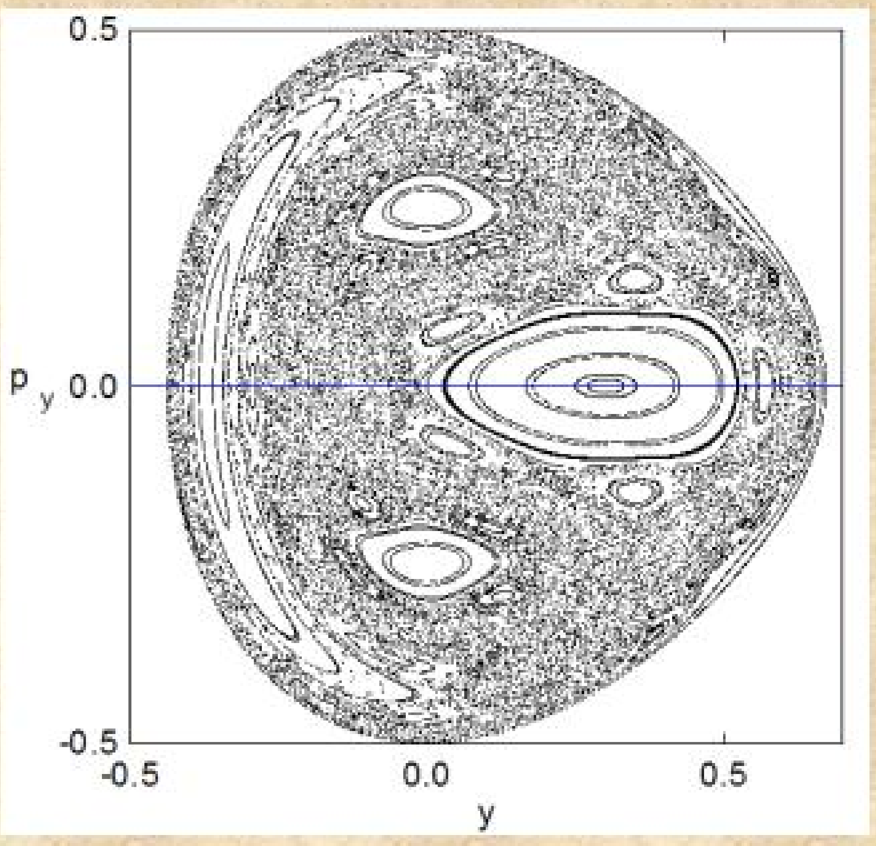
Chaotic orbit, x=0, y=-0.016, p_x=0.49974, p_y=0

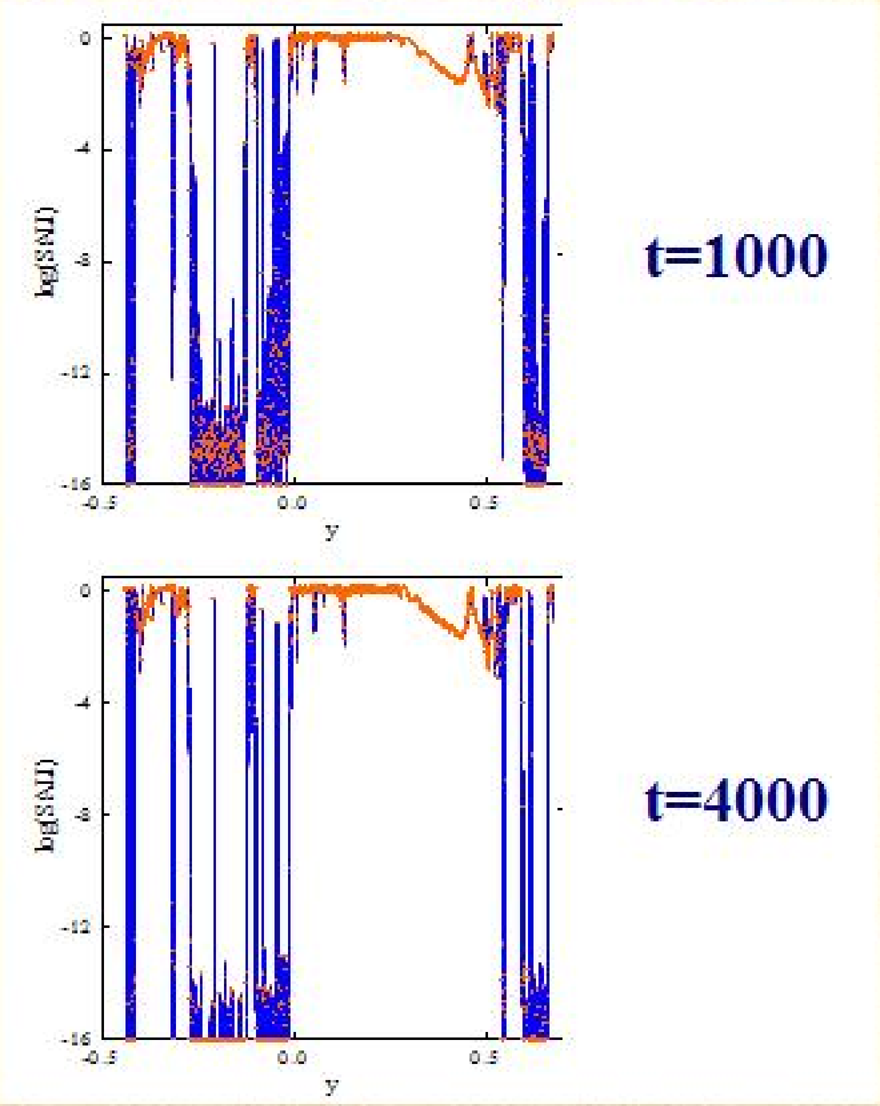
Chaotic orbit, x=0, y=-0.01344, p_x=0.49982, p_y=0



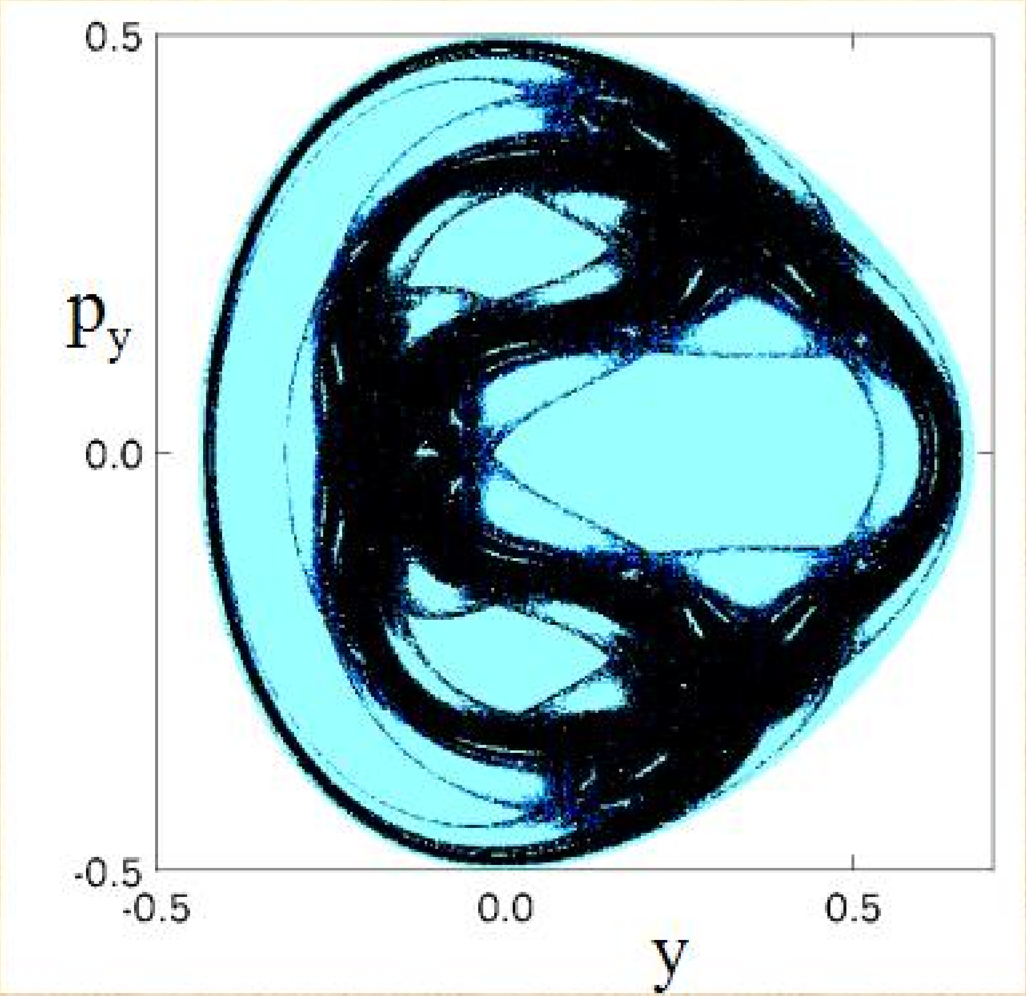


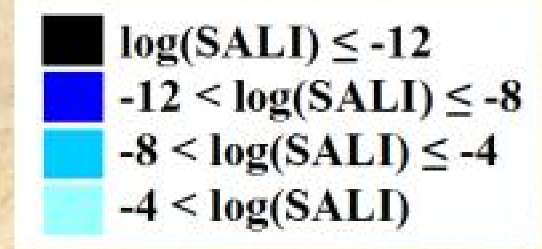
Applications – Hénon-Heils system





Applications – Hénon-Heils system





General References

- Lieberman A. J. & Lichtenberg M. A. , 1992, *Regular and Chaotic Dynamics,* Springer.
- Cvitanović P., Artuso R., Dahlqvist P., Mainieri R., Tanner G., Vattay G., Whelan N. & Wirzba A., 2003, *Chaos – Classical and Quantum,* http://www.nbi.dk/ChaosBook/